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Universal amplitude combinations for self-avoiding walks and polygons on directed lattices

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Abstract. Two directed lattices, the L lattice and Manhattan lattice, are studied. We have calculated exactly the number, the mean-square end-to-end distance, the mean-square radius of gyration and the mean-square distance of a monomer from the origin, for n-step self-avoiding walks on the L lattice and Manhattan lattice for up to 60 and 50 steps, respectively. We have also computed the number and mean-square radius of gyration for self-avoiding polygons on the L lattice and Manhattan lattice for up to 80 and 60 steps, respectively. We have estimated the critical amplitudes and our numerical results are consistent with the conjecture of universality for certain amplitude combinations. However, some amplitude combinations have values different from the corresponding values for undirected lattices.

1. Introduction

Self-avoiding walks (SAWs) and polygons (SAPs) on regular lattices are models of a polymer [1]. Several combinations of critical amplitudes for two-dimensional SAWs and SAPs are predicted to be lattice independent and then the predictions are confirmed for square, triangular, kagome and honeycomb lattices [2–10]. The purpose of the present paper is to study SAWs and SAPs on two directed lattices: the L lattice and Manhattan lattice. We want to check numerically whether the same amplitude combinations have the universal values for these two directed lattices, and whether these values are the same as the corresponding values for undirected lattices. We shall discuss the L lattice and Manhattan lattice in sections 2 and 3, respectively. A conclusion is given in section 4.

We are interested in the following functions:

(a) the generating function for SAWs $C(x) = \sum c_n x^n$ where c_n is the number of n-step SAWs;
(b) the generating function for SAPs $P(x) = \sum p_n x^n$ where p_n is the number of n-step SAPs;
(c) the mean-square end-to-end distance of n-step SAWs $\langle R_e^2 \rangle_n$;
(d) the mean-square radius of gyration of n-step polygons $\langle R^2 \rangle_n$;
(e) moments of the area of polygons of perimeter n, $\langle a^p \rangle_n$;
(f) the mean-square radius of gyration of n-step SAWs $\langle R_g^2 \rangle_n$; and
(g) the mean-square distance of a monomer from the origin of n-step SAWs $\langle R_m^2 \rangle_n$.

The asymptotic forms at large n are

$$\begin{aligned} c_n &= A\mu^n n^{\gamma-1}[1 + o(1)] \\ p_n &= B\mu^n n^{\alpha-3}[1 + o(1)] \\ \langle R_e^2\rangle_n &= Cn^{2\nu}[1 + o(1)] \\ \langle R^2\rangle_n &= Dn^{2\nu}[1 + o(1)] \\ \langle a^p\rangle_n &= [E^{(p)}n^{2\nu}]^p[1 + o(1)] \\ \langle R_g^2\rangle_n &= Fn^{2\nu}[1 + o(1)] \\ \langle R_m^2\rangle_n &= Gn^{2\nu}[1 + o(1)]. \end{aligned} \tag{1}$$

The exponents α, γ and ν depend only on the space dimensionality. The connective constant μ and the critical amplitudes $A, B, \ldots, G$ vary from lattice to lattice. Exact values of the exponents for two-dimensional lattices are known [11, 12]:

$$\gamma = \tfrac{43}{32} \qquad \alpha = \tfrac{1}{2} \qquad \nu = \tfrac{3}{4}. \tag{2}$$

Cardy and Saleur [5] used the c-theorem in conformal theory to prove that the amplitude ratios F/C and G/C are universal. For loose-packed and directed lattices, p_n is non-zero only if n is divisible by an integer σ. For the square lattice we have $\sigma = 2$. For the L and Manhattan lattices, we have $\sigma = 4$. Cardy and Guttmann [4] proved that

$$BD = \frac{5}{32\pi^2}\sigma a_0 \tag{3}$$

where a_0 is the area per lattice site. For example, we have $a_0 = 1$ for the square, L and Manhattan lattices. Camacho and Fisher [2] argued non-rigorously that the ratio $E^{(1)}/D$ should be universal. Privman and Redner [8] proved that $BC/\sigma a_0$ is universal.

2. The L lattice

The L lattice is an oriented square lattice in which each step must be followed by a step perpendicular to the preceding one. Grassberger calculated the SAWs on the L lattice up to 44 steps [13]. Guttmann analysed the series given by Grassberger to obtain [14]

$$1.5353 < \mu \approx 1.565\,75 \pm 0.0005 < 1.5986. \tag{4}$$

Enting and Guttmann calculated the number of SAPs up to 48 steps and their estimated values of the connective constant is [15]

$$\mu = 1.5657 \pm 0.0019. \tag{5}$$

Cardy and Guttmann analysed the SAW and SAP series to estimate the following amplitudes [4]:

$$A = 1.05 \qquad B = 2.47 \qquad C = 0.67 \tag{6}$$

they also predicted from the universal amplitude relations that

$$D = 0.049 \qquad E^{(1)} = 0.12 \qquad F = 0.095 \qquad G = 0.30. \tag{7}$$

However, they obtained

$$BC/\sigma a_0 = 0.41 \tag{8}$$

Table 1. Exact enumeration results for the number and mean-square radius of gyration for self-avoiding walks on the L lattice.

n	c_n	$(n+1)^2 c_n \langle R_g^2 \rangle_n$	n	c_n	$(n+1)^2 c_n \langle R_g^2 \rangle_n$
1	2	2	31	3 784 344	61 646 002 048
2	4	16	32	5 973 988	108 662 821 952
3	8	80	33	9 447 880	191 066 174 728
4	12	272	34	14 950 796	334 971 978 288
5	20	852	35	23 658 540	585 460 736 528
6	32	2 368	36	37 321 752	1 018 731 614 112
7	52	6 176	37	58 965 260	1 769 220 152 140
8	84	15 136	38	93 206 864	3 065 395 414 656
9	136	35 560	39	147 333 080	5 298 085 219 776
10	220	80 400	40	232 286 272	9 123 669 000 960
11	356	176 512	41	366 692 264	15 687 206 354 376
12	564	373 840	42	579 112 916	26 921 088 777 904
13	904	779 688	43	914 596 512	46 106 301 315 280
14	1 448	1 595 136	44	1 441 245 516	78 723 840 040 816
15	2 320	3 210 752	45	2 273 628 572	134 245 380 235 420
16	3 684	6 340 096	46	3 588 043 588	228 562 054 738 176
17	5 872	12 388 080	47	5 662 429 392	388 485 568 975 328
18	9 376	23 946 624	48	8 919 182 048	658 604 479 678 592
19	14 960	45 799 952	49	14 062 270 852	1 115 355 797 908 228
20	23 688	86 461 152	50	22 177 952 820	1 886 348 118 582 160
21	37 652	162 175 188	51	34 978 038 152	3 185 726 186 540 816
22	59 912	302 007 488	52	55 075 045 684	5 368 243 112 467 376
23	95 316	558 384 256	53	86 790 081 504	9 037 865 721 667 584
24	150 744	1 022 747 136	54	136 805 586 172	15 198 607 677 316 512
25	239 080	1 864 920 904	55	215 648 320 416	25 527 503 187 173 504
26	379 528	3 383 295 520	56	339 439 146 792	42 793 397 496 417 216
27	602 424	6 106 057 328	57	534 676 068 628	71 682 175 567 950 804
28	951 788	10 942 394 544	58	842 411 119 744	119 954 688 862 419 200
29	1 507 136	19 545 509 120	59	1 327 290 819 784	200 521 426 267 165 984
30	2 388 252	34 779 997 888	60	2 088 595 659 908	334 635 049 824 093 072

which is much larger than the corresponding value of 0.2168 for the square lattice or 0.2167 for the triangular lattice. In fact, the motivation of our present work is to check whether the directed lattices have the same values of amplitude combinations as the unoriented lattices.

We have calculated the number, mean-square radius of gyration, end-to-end distance and distance of a monomer from the origin, of SAWs on the L lattice up to 60 steps and give the results in tables 1 and 2. The perimeter and area generating function of SAPs on a two-dimensional lattice is defined as follows:

$$G(x, y) = \sum_{n,m} p(n, m)\, x^n y^m \qquad (9)$$

where $p(n, m)$ is the number of SAPs per lattice site with perimeter n and area (m). It is obvious that

$$p_n = \sum_m p(n, m). \qquad (10)$$

Table 2. Exact enumeration results for the mean-square end-to-end distance and the mean-square distance of a monomer from the origin for self-avoiding walks on the L lattice.

n	$c_n\langle R_e^2\rangle_n$	$(n+1)c_n\langle R_m^2\rangle_n$	n	$c_n\langle R_e^2\rangle_n$	$(n+1)c_n\langle R_m^2\rangle_n$
1	2	2	31	449 323 128	6 148 984 320
2	8	12	32	745 074 368	10 516 270 560
3	24	48	33	1 233 312 840	17 948 283 912
4	64	144	34	2 038 163 096	30 562 926 564
5	148	388	35	3 363 086 588	51 923 408 608
6	320	960	36	5 541 216 608	87 938 601 648
7	660	2 240	37	9 117 304 380	148 695 243 036
8	1 312	4 976	38	14 981 791 904	250 974 455 376
9	2 536	10 664	39	24 588 187 512	422 821 868 480
10	4 792	22 164	40	40 306 923 456	710 524 131 104
11	8 900	44 976	41	65 999 745 128	1 192 452 801 704
12	16 288	88 864	42	107 955 481 768	1 998 321 961 116
13	29 432	173 000	43	176 404 153 520	3 343 726 455 824
14	52 624	331 832	44	287 973 937 568	5 583 073 125 312
15	93 264	628 384	45	469 669 315 036	9 312 344 540 460
16	163 968	1 173 184	46	765 329 695 112	15 513 725 521 084
17	286 224	2 170 704	47	1 246 059 877 232	25 812 118 349 856
18	496 576	3 982 112	48	2 027 110 718 784	42 869 630 996 448
19	856 864	7 246 320	49	3 295 151 177 924	71 137 041 821 956
20	1 471 264	13 062 352	50	5 352 417 980 264	117 922 552 162 876
21	2 514 724	23 416 292	51	8 687 891 859 800	195 268 770 511 952
22	4 281 040	41 740 888	52	14 092 227 386 080	322 848 085 150 496
23	7 261 492	74 001 184	53	22 843 071 461 648	533 387 387 329 792
24	12 275 520	130 318 240	54	37 004 342 778 360	880 455 928 609 316
25	20 686 408	228 624 552	55	59 907 952 955 648	1 452 023 849 738 976
26	34 761 552	399 495 352	56	96 930 117 451 392	2 391 432 546 100 224
27	58 261 064	695 335 344	57	156 740 966 265 044	3 936 117 981 092 276
28	97 408 704	1 204 213 744	58	253 318 425 062 144	6 473 668 062 657 856
29	162 485 056	2 079 798 944	59	409 185 079 983 880	10 638 665 639 687 328
30	270 466 424	3 581 422 340	60	660 615 491 820 480	17 462 815 693 961 296

We have calculated $p(n,m)$ on the L lattice up to $n = 72$ and the results are given in tables 3–5. We computed p_n and the mean-square radius of gyration of n-step polygons $\langle R^2\rangle_n$ up to $n = 80$, and the moment of area of n-step polygons $\langle a\rangle_n$ up to $n = 72$ on the L lattice and the results are shown in table 6.

We use the method of differential approximants [16] to estimate the connective constant and the amplitudes. The results are:

$$
\begin{array}{lll}
\mu = 1.565\,90 \pm 0.000\,04 & A = 1.049 \pm 0.002 & B = 2.57 \pm 0.02 \\
C = 0.673 \pm 0.003 & D = 0.0487 \pm 0.0003 & E^{(1)} = 0.123 \pm 0.002 \\
E^{(2)} = 0.126 \pm 0.002 & E^{(3)} = 0.130 \pm 0.002 & E^{(4)} = 0.134 \pm 0.002 \\
E^{(5)} = 0.137 \pm 0.002 & E^{(6)} = 0.141 \pm 0.002 & E^{(7)} = 0.144 \pm 0.002 \\
E^{(8)} = 0.147 \pm 0.002 & E^{(9)} = 0.151 \pm 0.002 & E^{(10)} = 0.154 \pm 0.002 \\
F = 0.0933 \pm 0.002 & G = 0.295 \pm 0.002. &
\end{array}
\qquad (11)
$$

Table 3. Number of polygons on the L lattice with perimeter $n \leqslant 56$.

n, m	$p(n, m)$	n, m	$p(n, m)$	n, m	$p(n, m)$	n, m	$p(n, m)$	n, m	$p(n, m)$
4, 1	1	36, 33	151	44, 47	1 664	52, 27	292	56, 36	200 184
12, 5	1	36, 35	68	44, 49	894	52, 29	4 401	56, 38	339 178
16, 8	2	36, 37	22	44, 51	426	52, 31	21 628	56, 40	469 384
20, 9	2	36, 39	6	44, 53	187	52, 33	54 460	56, 42	575 758
20, 11	6	36, 41	1	44, 55	68	52, 35	91 166	56, 44	648 944
20, 13	1	40, 20	24	44, 57	22	52, 37	121 752	56, 46	684 978
24, 12	8	40, 22	344	44, 59	6	52, 39	142 972	56, 48	697 128
24, 14	18	40, 24	1184	44, 61	1	52, 41	154 750	56, 50	689 882
24, 16	8	40, 26	1904	48, 24	32	52, 43	157 926	56, 52	662 584
24, 18	2	40, 28	2160	48, 26	912	52, 45	156 182	56, 54	620 046
28, 13	2	40, 30	2246	48, 28	5 556	52, 47	149 600	56, 56	565 216
28, 15	28	40, 32	2120	48, 30	15 008	52, 49	137 268	56, 58	500 224
28, 17	55	40, 34	1786	48, 32	24 762	52, 51	121 564	56, 60	429 296
28, 19	40	40, 36	1280	48, 34	31 592	52, 53	103 389	56, 62	357 294
28, 21	22	40, 38	824	48, 36	35 354	52, 55	84 000	56, 64	286 064
28, 23	6	40, 40	448	48, 38	36 872	52, 57	64 862	56, 66	220 464
28, 25	1	40, 42	218	48, 40	36 542	52, 59	47 774	56, 68	163 152
32, 16	16	40, 44	88	48, 42	34 680	52, 61	33 133	56, 70	115 886
32, 18	96	40, 46	30	48, 44	31 322	52, 63	21 768	56, 72	78 688
32, 20	170	40, 48	8	48, 46	26 416	52, 65	13 500	56, 74	51 110
32, 22	160	40, 50	2	48, 48	21 132	52, 67	7 898	56, 76	31 632
32, 24	130	44, 21	2	48, 50	15 792	52, 69	4 314	56, 78	18 718
32, 26	72	44, 23	168	48, 52	10 992	52, 71	2 216	56, 80	10 488
32, 28	30	44, 25	1402	48, 54	7 072	52, 73	1 054	56, 82	5 576
32, 30	8	44, 27	4192	48, 56	4 264	52, 75	470	56, 84	2 800
32, 32	2	44, 29	6807	48, 58	2 344	52, 77	187	56, 86	1 332
36, 17	2	44, 31	8208	48, 60	1 196	52, 79	68	56, 88	584
36, 19	80	44, 33	8806	48, 62	560	52, 81	22	56, 90	238
36, 21	336	44, 35	8878	48, 64	238	52, 83	6	56, 92	88
36, 23	552	44, 37	8405	48, 66	88	52, 85	1	56, 94	30
36, 25	586	44, 39	7288	48, 68	30	56, 28	40	56, 96	8
36, 27	572	44, 41	5840	48, 70	8	56, 30	1 980	56, 98	2
36, 29	454	44, 43	4200	48, 72	2	56, 32	19 928		
36, 31	298	44, 45	2794	52, 25	2	56, 34	83 506		

It follows from (11) that

$$BC/\sigma a_0 = 0.432 \pm 0.006 \qquad 32\pi^2 BD/5\sigma a_0 = 1.976 \pm 0.030 \tag{12}$$

which are twice as large as the corresponding values for unoriented lattices. However, the amplitude ratio

$$E^{(1)}/D = 2.526 \pm 0.060 \tag{13}$$

is close to the corresponding values of 2.515 for the square lattice and 2.529 for the triangular lattice given by Cardy and Guttmann [4].

The amplitude ratios F/C and G/C can be determined from equation (11). These ratios can be obtained directly with greater accuracy by the method of Meir [17] as follows. We construct two new series

$$R(x) = \sum r_n x^n \qquad S(x) = \sum s_n x^n \tag{14}$$

Table 4. Number of polygons on the L lattice with perimeter $n = 60, 64, 68$.

n, m	$p(n, m)$	n, m	$p(n, m)$	n, m	$p(n, m)$	n, m	$p(n, m)$
60, 29	2	60, 105	187	64, 96	482 892	68, 73	57 835 152
60, 31	452	60, 107	68	64, 98	315 344	68, 75	53 829 520
60, 33	11 354	60, 109	22	64, 100	199 628	68, 77	49 464 980
60, 35	86 576	60, 110	6	64, 102	122 272	68, 79	44 857 890
60, 37	321 720	60, 111	1	64, 104	72 476	68, 81	40 137 349
60, 39	744 658			64, 106	41 392	68, 83	35 403 848
60, 41	1 274 726	64, 32	48	64, 108	22 794	68, 85	30 744 538
60, 43	1 814 756	64, 34	3 712	64, 110	12 048	68, 87	26 276 798
60, 45	2 306 542	64, 36	58 876	64, 112	6 116	68, 89	22 084 728
60, 47	2 698 796	64, 38	365 760	64, 114	2 960	68, 91	18 229 528
60, 49	2 955 435	64, 40	1 240 156	64, 116	1 360	68, 93	14 761 878
60, 51	3 099 460	64, 42	2 797 848	64, 118	584	68, 95	11 722 186
60, 53	3 148 886	64, 44	4 834 380	64, 120	238	68, 97	9 117 699
60, 55	3 115 902	64, 46	7 049 736	64, 122	88	68, 99	6 942 428
60, 57	3 018 196	64, 48	9 222 726	64, 124	30	68, 101	5 169 736
60, 59	2 865 104	64, 50	11 136 152	64, 126	8	68, 103	3 762 938
60, 61	2 660 242	64, 52	12 620 384	64, 128	2	68, 105	2 675 153
60, 63	2 420 116	64, 54	13 658 048			68, 107	1 857 128
60, 65	2 158 271	64, 56	14 256 684	68, 33	2	68, 109	1 257 308
60, 67	1 879 280	64, 58	14 468 280	68, 35	648	68, 111	829 924
60, 69	1 596 256	64, 60	14 382 652	68, 37	25 080	68, 113	533 333
60, 71	1 319 680	64, 62	14 044 864	68, 39	286 088	68, 115	333 668
60, 73	1 060 697	64, 64	13 463 042	68, 41	1 516 868	68, 117	202 970
60, 75	826 960	64, 66	12 696 336	68, 43	4 789 792	68, 119	119 920
60, 77	624 700	64, 68	11 791 216	68, 45	10 599 164	68, 121	68 617
60, 79	456 212	64, 70	10 773 264	68, 47	18 471 866	68, 123	38 040
60, 81	322 413	64, 72	9 682 166	68, 49	27 513 458	68, 125	20 364
60, 83	219 768	64, 74	8 549 464	68, 51	36 898 540	68, 127	10 514
60, 85	144 374	64, 76	7 409 356	68, 53	45 754 550	68, 129	5 202
60, 87	91 286	64, 78	6 294 560	68, 55	53 395 762	68, 131	2 468
60, 89	55 641	64, 80	5 234 620	68, 57	59 510 695	68, 133	1 106
60, 91	32 504	64, 82	4 253 360	68, 59	63 858 564	68, 135	470
60, 93	18 204	64, 84	3 376 708	68, 61	66 454 224	68, 137	187
60, 95	9 738	64, 86	2 614 840	68, 63	67 607 750	68, 139	68
60, 97	4 978	64, 88	1 972 238	68, 65	67 542 967	68, 141	22
60, 99	2 408	64, 90	1 447 248	68, 67	66 346 336	68, 143	6
60, 101	1 106	64, 92	1 033 568	68, 69	64 231 106	68, 145	1
60, 103	470	64, 94	716 968	68, 71	61 354 036		

where

$$r_n = \langle R_g^2 \rangle_n / \langle R_e^2 \rangle_n \qquad s_n = \langle R_m^2 \rangle_n / \langle R_e^2 \rangle_n.$$

For large n we have

$$r_n \approx F/C \qquad s_n \approx G/C. \tag{15}$$

These two generating functions have a simple pole at $x = 1$, and the residues at this pole are the amplitude ratios. We construct Padé approximants to $(1-x)R(x)$ and $(1-x)S(x)$ and then evaluate them at $x = 1$ to estimate the amplitude ratios of interest. The results are

$$F/C = 0.139 \pm 0.001 \qquad G/C = 0.439 \pm 0.001. \tag{16}$$

Table 5. Number of polygons on the L lattice with perimeter $n = 72$.

m	$p(72, m)$	m	$p(72, m)$	m	$p(72, m)$
36	56	80	269 791 664	124	2 591 960
38	6 300	82	252 088 066	126	1 706 804
40	149 104	84	233 151 528	128	1 098 688
42	1 329 030	86	213 406 090	130	691 006
44	6 216 600	88	193 182 984	132	424 088
46	18 551 888	90	172 916 852	134	253 898
48	40 437 776	92	152 991 976	136	147 920
50	71 022 664	94	133 705 292	138	83 844
52	107 788 072	96	115 336 960	140	46 128
54	147 794 916	98	98 149 900	142	24 602
56	187 639 752	100	82 331 936	144	12 672
58	224 548 252	102	68 037 760	146	6 300
60	256 789 680	104	55 347 088	148	2 992
62	282 761 574	106	44 289 818	150	1 360
64	301 658 896	108	34 841 208	152	584
66	313 965 386	110	26 932 254	154	238
68	320 332 800	112	20 443 120	156	88
70	321 281 238	114	15 231 574	158	30
72	317 693 176	116	11 130 368	160	8
74	310 209 836	118	7 974 358	162	2
76	299 339 760	120	5 598 928		
78	285 693 424	122	3 850 916		

Table 6. Exact enumeration results for the number, mean-square radius of gyration, and moment of area for polygons on the L lattice.

n	p_n	$n^2 p_n \langle R^2 \rangle_n$	$p_n \langle a \rangle_n$
4	1	8	1
12	1	264	5
16	2	1 536	16
20	9	14 216	97
24	36	108 928	512
28	154	803 984	2 766
32	684	5 744 128	15 040
36	3 128	39 983 744	82 256
40	14 666	273 051 648	452 596
44	70 258	1 837 607 824	2 505 366
48	342 766	12 223 570 944	13 944 632
52	1 698 625	80 542 524 808	77 999 517
56	8 532 410	526 542 581 504	438 228 204
60	43 368 153	3 419 485 970 888	2 471 918 185
64	222 729 492	22 081 539 551 744	13 993 276 592
68	1 154 455 161	141 897 671 721 864	79 470 121 673
72	6 033 034 032	907 968 336 775 424	452 641 358 912
76	31 760 434 883	5 788 138 103 404 184	
80	168 311 948 410	36 776 052 223 853 824	

From a Monte Carlo study of SAWs on the square lattice, Caracciolo *et al* found that [3]

$$F/C = 0.140\,26 \pm 0.000\,11 \qquad G/C = 0.439\,62 \pm 0.000\,33 \tag{17}$$

which are in excellent agreement with our estimations.

Table 7. Exact enumeration results for the number and mean-square radius of gyration for self-avoiding walks on the Manhattan lattice.

n	c_n	$(n+1)^2 c_n \langle R_g^2 \rangle_n$	n	c_n	$(n+1)^2 c_n \langle R_g^2 \rangle_n$
1	2	2	26	4 457 332	50 485 490 336
2	4	20	27	7 835 308	100 451 116 128
3	8	112	28	13 687 192	198 332 595 928
4	14	456	29	24 008 300	390 733 133 148
5	26	1 626	30	42 118 956	766 795 102 184
6	48	5 136	31	73 895 808	1 499 229 115 184
7	88	14 928	32	129 012 260	2 913 859 108 256
8	154	39 952	33	225 966 856	5 653 840 726 744
9	278	103 702	34	395 842 772	10 937 332 416 456
10	500	258 692	35	693 470 658	21 097 016 063 128
11	900	625 440	36	1 210 093 142	40 501 312 004 980
12	1 576	1 454 304	37	2 117 089 488	77 650 976 678 800
13	2 806	3 338 454	38	3 704 400 974	148 518 976 628 104
14	4 996	7 519 992	39	6 482 222 366	283 404 570 480 304
15	8 894	16 662 352	40	11 306 267 980	538 703 646 466 264
16	15 564	36 093 832	41	19 762 622 872	1 022 900 599 888 312
17	27 538	77 645 650	42	34 547 967 024	1 938 498 991 877 448
18	48 726	165 180 100	43	60 398 557 080	3 666 645 591 067 880
19	86 212	347 863 040	44	105 304 986 402	6 913 108 881 059 252
20	150 792	721 871 816	45	183 929 200 768	13 022 667 880 888 576
21	265 730	1 491 459 762	46	321 291 900 388	24 491 661 962 488 824
22	468 342	3 058 946 592	47	561 272 370 806	45 987 976 638 118 936
23	825 462	6 231 171 176	48	978 233 485 056	86 116 824 865 781 960
24	1 442 866	12 562 042 104	49	1 707 553 933 232	161 144 603 202 059 856
25	2 535 802	25 248 996 858	50	2 980 910 982 162	301 123 764 733 014 316

3. The Manhattan lattice

In the Manhattan lattice, adjacent rows (columns) have antiparallel directions, similar to the traffic pattern in Manhattan. Kasteleyn proved that the Manhattan lattice is the covering lattice of the L lattice [18]. Malakis calculated the SAWs up to 28 steps [19]. Guttmann analysed the series given by Malakis to obtain [14]

$$1.6336 < \mu \approx 1.7340 \pm 0.0009 < 1.7912. \tag{18}$$

Enting and Guttmann calculated the number of SAPs up to 48 steps and their estimated values of the connective constant is [15]

$$\mu = 1.7328 \pm 0.0005. \tag{19}$$

Cardy and Guttmann analysed the SAW and SAP series to estimate the following amplitudes [4]:

$$A = 0.89 \qquad B = 2.5 \qquad C = 0.73 \tag{20}$$

they predicted from the universal amplitude relations that

$$D = 0.053 \qquad E^{(1)} = 0.13 \qquad F = 0.10 \qquad G = 0.32. \tag{21}$$

It follows from the above data that [4]

$$BC/\sigma a_0 = 0.46. \tag{22}$$

Table 8. Exact enumeration results for the mean-square end-to-end distance and the mean-square distance of a monomer from the origin for self-avoiding walks on the Manhattan lattice.

n	$c_n\langle R_e^2\rangle_n$	$(n+1)c_n\langle R_m^2\rangle_n$	n	$c_n\langle R_e^2\rangle_n$	$(n+1)c_n\langle R_m^2\rangle_n$
1	2	2	26	467 431 032	5 669 646 276
2	12	16	27	865 237 516	10 885 908 760
3	40	72	28	1 597 970 936	20 789 214 036
4	112	248	29	2 944 886 876	39 625 084 064
5	282	754	30	5 416 048 776	75 303 431 932
6	656	2 064	31	9 942 306 720	142 713 087 288
7	1 464	5 304	32	18 219 575 032	269 345 135 040
8	3 168	12 816	33	33 332 660 040	507 593 255 016
9	6 678	30 102	34	60 885 660 048	954 393 225 124
10	13 788	68 544	35	111 051 768 370	1 790 623 853 236
11	28 036	152 464	36	202 274 103 896	3 348 461 256 636
12	56 264	329 812	37	367 945 659 312	6 254 392 854 296
13	111 638	705 254	38	668 465 444 552	11 660 915 285 586
14	219 336	1 486 052	39	1 213 006 651 646	21 703 488 659 224
15	427 422	3 093 064	40	2 198 692 791 272	40 286 102 087 068
16	827 040	6 336 372	41	3 981 080 342 824	74 709 808 414 308
17	1 590 082	12 900 166	42	7 200 946 110 672	138 342 005 385 528
18	3 039 264	26 037 742	43	13 012 408 511 720	255 808 158 198 948
19	5 780 100	52 161 584	44	23 492 347 156 296	471 955 224 328 732
20	10 943 728	103 424 412	45	42 374 959 452 432	870 073 114 312 284
21	20 634 498	204 269 370	46	76 369 410 144 560	1 602 042 428 723 764
22	38 756 688	401 171 114	47	137 523 678 800 774	2 946 303 202 056 852
23	72 547 590	783 894 612	48	247 456 917 611 328	5 408 264 157 499 420
24	135 381 160	1 520 685 216	49	444 934 775 413 856	9 921 133 505 696 948
25	251 901 866	2 942 140 126	50	799 425 383 404 744	18 180 744 569 257 394

It is pointed out by Cardy and Guttmann [4] that the polygon counts for Manhattan lattice polygons obtained by Enting and Guttmann [15] should be divided by two in order that the correct normalization per site be retained.

We have calculated the number, mean-square radius of gyration, end-to-end distance and distance of a monomer from the origin, of SAWs on the Manhattan lattice up to 50 steps and give the results in tables 7 and 8. We have computed the number, mean-square radius of gyration and moment of area for SAPs up to 60 steps and the results are shown in tables 9–12.

We use the method of differential approximants [16] to estimate the connective constant and the amplitudes. The results are:

$$\begin{array}{lll} \mu = 1.7335 \pm 0.0003 & A = 0.863 \pm 0.003 & B = 2.51 \pm 0.02 \\ C = 0.71 \pm 0.02 & D = 0.051 \pm 0.002 & E^{(1)} = 0.129 \pm 0.002 \\ E^{(2)} = 0.133 \pm 0.002 & E^{(3)} = 0.137 \pm 0.002 & E^{(4)} = 0.142 \pm 0.002 \\ E^{(5)} = 0.147 \pm 0.002 & E^{(6)} = 0.152 \pm 0.002 & E^{(7)} = 0.157 \pm 0.002 \\ E^{(8)} = 0.162 \pm 0.002 & E^{(9)} = 0.167 \pm 0.002 & E^{(10)} = 0.172 \pm 0.002 \\ F = 0.100 \pm 0.002 & G = 0.315 \pm 0.002. & \end{array} \tag{23}$$

It follows from (23) that

$$\begin{array}{ll} BC/\sigma a_0 = 0.446 \pm 0.016 & 32\pi^2 BD/5\sigma a_0 = 2.02 \pm 0.10 \\ E^{(1)}/D = 2.53 \pm 0.14. & \end{array} \tag{24}$$

Table 9. Number of polygons on the Manhattan lattice with perimeter $n \leqslant 48$.

n, m	$2p(n, m)$	n, m	$2p(n, m)$	n, m	$2p(n, m)$	n, m	$2p(n, m)$
4, 1	1	32, 39	1 128	40, 63	10 568	44, 121	1
8, 3	2	32, 43	528	40, 67	5 560	48, 23	3 150 362
12, 5	6	32, 47	230	40, 71	2 760	48, 27	7 387 128
12, 9	1	32, 51	88	40, 75	1 304	48, 31	10 850 144
16, 7	22	32, 55	30	40, 79	576	48, 35	12 466 608
16, 11	8	32, 59	8	40, 83	238	48, 39	12 421 706
16, 15	2	32, 63	2	40, 87	88	48, 43	11 207 656
20, 9	87	36, 17	31 581	40, 91	30	48, 47	9 453 084
20, 13	52	36, 21	50 056	40, 95	8	48, 51	7 554 640
20, 17	22	36, 25	52 644	40, 99	2	48, 55	5 790 492
20, 21	6	36, 29	43 910	44, 21	672 390	48, 59	4 277 952
20, 25	1	36, 33	32 313	44, 25	1 406 027	48, 63	3 061 620
24, 11	364	36, 37	21 468	44, 29	1 872 928	48, 67	2 125 600
24, 15	304	36, 41	13 352	44, 33	1 963 444	48, 71	1 436 140
24, 19	182	36, 45	7 714	44, 37	1 796 424	48, 75	943 696
24, 23	80	36, 49	4 210	44, 41	1 491 541	48, 79	604 180
24, 27	30	36, 53	2 148	44, 45	1 160 480	48, 83	376 568
24, 31	8	36, 57	1 026	44, 49	854 344	48, 87	228 642
24, 35	2	36, 61	454	44, 53	602 290	48, 91	134 952
28, 13	1574	36, 65	187	44, 57	407 745	48, 95	77 512
28, 17	1706	36, 69	68	44, 61	266 220	48, 99	43 152
28, 21	1288	36, 73	22	44, 65	167 634	48, 103	23 314
28, 25	758	36, 77	6	44, 69	102 136	48, 107	12 144
28, 29	390	36, 81	1	44, 73	60 021	48, 111	6 104
28, 33	171	40, 19	144 880	44, 77	34 080	48, 115	2 936
28, 37	68	40, 23	266 240	44, 81	18 652	48, 119	1 352
28, 41	22	40, 27	317 716	44, 85	9 822	48, 123	584
28, 45	6	40, 31	299 848	44, 89	4 950	48, 127	238
28, 49	1	40, 35	248 552	44, 93	2 388	48, 131	88
32, 15	6986	40, 39	186 864	44, 97	1 090	48, 135	30
32, 19	9312	40, 43	131 830	44, 101	470	48, 139	8
32, 23	8442	40, 47	87 504	44, 105	187	48, 143	2
32, 27	6048	40, 51	55 412	44, 109	68		
32, 31	3832	40, 55	33 392	44, 113	22		
32, 35	2152	40, 59	19 258	44, 117	6		

The amplitude ratios F/C and G/C are determined by the method of Meir [17] and the results are

$$F/C = 0.141 \pm 0.002 \qquad G/C = 0.441 \pm 0.003. \tag{25}$$

4. Conclusion

The SAWs and SAPs on two directed lattices, the L lattice and the Manhattan lattice, are studied numerically. The critical amplitudes are estimated. The estimated values of three amplitude ratios,

$$F/C \qquad G/C \qquad E^{(1)}/D \tag{26}$$

Table 10. Number of polygons on the Manhattan lattice with perimeter $n = 52, 56$.

n, m	$2p(n, m)$	n, m	$2p(n, m)$	n, m	$2p(n, m)$
52, 25	14 877 317	52, 133	10 526	56, 91	53 726 724
52, 29	38 664 580	52, 137	5 190	56, 95	38 014 560
52, 33	62 013 464	52, 141	2 452	56, 99	26 452 810
52, 37	77 322 400	52, 145	1 106	56, 103	18 101 664
52, 41	83 079 096	52, 149	470	56, 107	12 185 456
52, 45	80 573 684	52, 153	187	56, 111	8 066 504
52, 49	72 815 862	52, 157	68	56, 115	5 251 504
52, 53	62 316 234	52, 161	22	56, 119	3 360 400
52, 57	51 137 585	52, 165	6	56, 123	2 113 428
52, 61	40 510 592	52, 169	1	56, 127	1 305 040
52, 65	31 146 856			56, 131	791 056
52, 69	23 306 236	56, 27	70 726 936	56, 135	470 056
52, 73	17 021 749	56, 31	201 788 632	56, 139	273 706
52, 77	12 145 484	56, 35	350 661 684	56, 143	155 832
52, 81	8 478 790	56, 39	470 795 992	56, 147	86 720
52, 85	5 793 768	56, 43	541 431 120	56, 151	47 008
52, 89	3 877 737	56, 47	559 955 960	56, 155	24 810
52, 93	2 541 460	56, 51	537 768 478	56, 159	12 688
52, 97	1 631 776	56, 55	488 245 688	56, 163	6 280
52, 101	1 025 626	56, 59	424 586 256	56, 167	2 984
52, 105	631 105	56, 63	356 452 264	56, 171	1 360
52, 109	379 716	56, 67	290 624 934	56, 175	584
52, 113	223 290	56, 71	230 943 312	56, 179	238
52, 117	128 136	56, 75	179 423 962	56, 183	88
52, 121	71 697	56, 79	136 508 192	56, 187	30
52, 125	39 004	56, 83	101 871 316	56, 191	8
52, 129	20 600	56, 87	74 632 856	56, 195	2

Table 11. Number of polygons on the Manhattan lattice with perimeter $n = 60$.

m	$2p(60, m)$	m	$2p(60, m)$	m	$2p(60, m)$
29	338 158 676	97	486 055 070	165	432 964
33	1 050 776 290	101	359 774 940	169	248 190
37	1 965 655 356	105	262 580 942	173	139 316
41	2 824 001 544	109	189 036 586	177	76 421
45	3 455 807 384	113	134 253 851	181	40 908
49	3 788 410 061	117	94 076 056	185	21 304
53	3 842 874 952	121	65 041 108	189	10 766
57	3 676 897 210	125	44 368 166	193	5 254
61	3 363 977 782	129	29 856 395	197	2 468
65	2 968 908 704	133	19 817 056	201	1 106
69	2 544 050 388	137	12 969 752	205	470
73	2 125 642 650	141	8 367 800	209	187
77	1 737 664 386	145	5 319 189	213	68
81	1 392 812 732	149	3 330 224	217	22
85	1 096 613 400	153	2 051 912	221	6
89	849 122 856	157	1 243 554	225	1
93	647 277 550	161	740 549		

appear to be universal and agree with the corresponding values for undirected lattices. However, the estimated values of two amplitude combinations

$$BC/\sigma a_0 \qquad 32\pi^2 BD/5\sigma a_0 \tag{27}$$

Table 12. Exact enumeration results for the number, mean-square radius of gyration and moment of area for polygons on the Manhattan lattice.

n	$2p_n$	$2n^2 p_n \langle R^2\rangle_n$	$2p_n\langle a\rangle_n$
4	1	8	1
8	2	192	6
12	7	2 680	39
16	32	32 256	272
20	168	360 768	1 984
24	970	3 867 072	14 990
28	5 984	40 304 768	116 152
32	38 786	411 841 792	917 534
36	261 160	4 146 385 216	7 358 056
40	1 812 630	41 267 794 496	59 725 714
44	12 895 360	406 958 302 208	489 628 168
48	93 638 634	3 982 960 196 352	4 047 259 878
52	691 793 872	38 736 273 555 840	33 689 244 776
56	5 186 869 122	374 713 379 107 776	282 111 290 118
60	39 388 514 522	3 608 070 750 706 768	2 374 646 644 818

are twice as large as the corresponding universal values for undirected lattices. It is interesting to note that these two combinations have the common factor B, which is the critical amplitude for the generating function of the number of SAPs.

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